Cyclic factorization numbers of finite groups

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Abstract

In this paper we introduce and study the concept of cyclic factorization number of a finite group G. By using the Möbius inversion formula and other methods involving the cyclic subgroup structure, this is explicitly computed for some important classes of finite groups.

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1 Introduction

Let G be a finite group, $L(G)$ be the subgroup lattice of G and $L_1(G)$ be the poset of cyclic subgroups of G. One of the most interesting positive integers that can be associated to G is the *factorization number* $F_2(G)$. This counts the number of all pairs $(H, K) \in L(G)^2$ satisfying $G = HK$ and has been investigated in many recent papers, such as [5, 12]. On the other hand, in our previous paper [9] (see also [11]) we defined the *subgroup commutativity degree* $sd(G)$ of G as the proportion of the number of ordered pairs $(H, K) \in L(G)^2$ such that $HK = KH$ by $|L(G)|^2$. These two quantities are closely connected. More precisely, we have

$$sd(G) = \frac{1}{|L(G)|^2} \sum_{H \leq G} F_2(H) \tag{1}$$

and

$$F_2(G)=\sum_{H\leq G} sd(H)|L(H)|^2\mu(H,G)\,, \tag{2}$$

by applying the well-known Möbius inversion formula. We also recall the following theorem due to P. Hall [2] (see also [3]), that permits us to compute explicitly the Möbius function of a finite p-group.

Theorem 1.1. *Let G be a finite p-group of order p^n. Then $\mu(G)=0$ unless G is elementary abelian, in which case we have $\mu(G)=(-1)^n p^{\binom{n}{2}}$.*

Then, in [14], we defined the *cyclic subgroup commutativity degree $csd(G)$* of G by replacing $L(G)$ with $L_1(G)$ in the definition of $sd(G)$. This also has a probabilistic significance, namely it measures the probability that two cyclic subgroups of G commute. So, it is naturally to introduce a new positive integer that corresponds to $csd(G)$ in the same way as $F_2(G)$ corresponds to $sd(G)$. In the following we will denote by $CF_2(G)$ the number of all pairs $(H,K)\in L_1(G)^2$ satisfying $G=HK$ and we will call it the *cyclic factorization number* of G. Its study is the purpose of the current paper.

The paper is organized as follows. Some basic properties of cyclic factorization numbers are presented in Section 2. Section 3 deals with the computation of $CF_2(G)$ for some classes of finite groups. In the final section some further research directions and a list of open problems are indicated.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [4, 7]. For subgroup lattice concepts we refer the reader to [6, 8, 13].

2 Basic properties of cyclic factorization numbers

Let G be a finite group. First of all, we remark that $CF_2(G)\leq F_2(G)$, and we have equality if and only if G is cyclic. Consequently, for such a group G the number $CF_2(G)$ is given by **Theorem 1.1.** of [5].

Proposition 2.1. *Let G be a finite cyclic group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then*

$$CF_2(G) = \Phi_1(n) = \prod_{i=1}^{k} (2\alpha_i + 1)\,.$$

Given two finite groups G and G', if $G \cong G'$ then $CF_2(G) = CF_2(G')$. By **Proposition 2.1.** we infer that the converse is not true. Also, we note that the weaker condition $L(G) \cong L(G')$ does not imply $CF_2(G) = CF_2(G')$, as shows the following elementary example.

Example 2.2. It is well-known that the subgroup lattices of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $G' = S_3$ are isomorphic. On the other hand, we can easily check that $CF_2(G) = 12 \neq 6 = CF_2(G')$.

By a direct calculation, one obtains

$$CF_2(S_3 \times \mathbb{Z}_2) = 12 \neq 18 = CF_2(S_3) CF_2(\mathbb{Z}_2)$$

and therefore in general we don't have

$$CF_2(G \times G') = CF_2(G) CF_2(G').$$

A sufficient condition in order to this equality holds is that G and G' be of coprime orders. This remark can naturally be extended to arbitrary finite direct products.

Proposition 2.3. *Let $(G_i)_{i=\overline{1,k}}$ be a family of finite groups having coprime orders. Then*

$$CF_2(\bigtimes_{i=1}^{k} G_i) = \prod_{i=1}^{k} CF_2(G_i)\,.$$

The following immediate consequence of **Proposition 2.3.** shows that the computation of $CF_2(G)$ for a finite nilpotent group G can be reduced to p-groups.

Corollary 2.4. *If G is a finite nilpotent group and $(G_i)_{i=\overline{1,k}}$ are the Sylow subgroups of G, then*

$$CF_2(G) = \prod_{i=1}^{k} CF_2(G_i)\,.$$

Remarks 2.5.

a) The conclusion of **Corollary 2.4.** fails if G is not nilpotent. For example, we have $CF_2(S_4) = CF_2(A_4) = 0$ even if $CF_2(S) \neq 0$ for every Sylow subgroup S of S_4 and A_4, respectively. Notice also that more can be said about such groups, namely $CF_2(S_n) = CF_2(A_n) = 0$ for all $n \geq 4$.

b) Let $p \geq 3$ be a prime and G be a finite p-group. It is well-known (see e.g. [4], I) that G can be written as a product of two cyclic subgroups if and only if it is metacyclic. This can be reformulated in the following nice way: "A finite p-group with $p \geq 3$ is metacyclic if and only if its cyclic factorization number is non-zero".

Finally, we observe that the connections between $CF_2(G)$ and $csd(G)$ are similar with (1) and (2), more exactly

$$csd(G) = \frac{1}{|L_1(G)|^2} \sum_{H \leq G} CF_2(H) \tag{3}$$

and

$$CF_2(G) = \sum_{H \leq G} csd(H)|L_1(H)|^2 \mu(H, G) \,. \tag{4}$$

The equality (4) is the main ingredient that will be used in Section 3 to calculate the cyclic factorization numbers of certain finite groups.

3 Cyclic factorization numbers for some classes of finite groups

As we already have seen, the computation of cyclic factorization numbers of finite abelian groups is reduced to p-groups. By the fundamental theorem of finitely generated abelian groups, such a group is of type

$$G \cong \bigtimes_{i=1}^{k} \mathbb{Z}_{p^{\alpha_i}} \,,$$

where $1 \leq \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_k$. Recall that G possesses a unique maximal elementary abelian subgroup $M \cong \mathbb{Z}_p^k$. Moreover, all elementary abelian subgroups of G are contained in M. Notice also that we have $csd(H) = 1$, for every $H \leq G$. We are now able to compute explicitly $CF_2(G)$.

Theorem 3.1. *The cyclic factorization number of the finite abelian p-group* $G \cong \bigtimes_{i=1}^{k} \mathbb{Z}_{p^{\alpha_i}}$, $1 \leq \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_k$, *is given by the following equality:*

$$CF_2(G) = \begin{cases} 2\alpha_1{+}1, & \text{if } k = 1 \\ p^{2\alpha_1 - 1}\left[(2\alpha_2{-}2\alpha_1{+}1)p{-}2\alpha_2{+}2\alpha_1{+}1\right], & \text{if } k = 2 \\ 0, & \text{if } k \geq 3\,. \end{cases}$$

Proof. By using (4) and **Theorem 1.1.**, one obtains

$$(5)\quad CF_2(G) = \sum_{H \leq G} |L_1(H)|^2 \mu(H,G) = \sum_{H \leq G} |L_1(G/H)|^2 \mu(H) =$$

$$= \sum_{H \leq M} |L_1(G/H)|^2 \mu(H) = \sum_{i=0}^{k} \sum_{H \leq M,\, |H| = p^i} |L_1(G/H)|^2 (-1)^i p^{\binom{i}{2}}\,.$$

For $k = 1$ we have

$$CF_2(G) = |L_1(\mathbb{Z}_{p^{\alpha_1}})|^2 - |L_1(\mathbb{Z}_{p^{\alpha_1 - 1}})|^2 = (\alpha_1 + 1)^2 - \alpha_1^2 = 2\alpha_1 + 1\,,$$

while for $k = 2$ we have

$$CF_2(G) = |L_1(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}})|^2 - p|L_1(\mathbb{Z}_{p^{\alpha_1 - 1}} \times \mathbb{Z}_{p^{\alpha_2}})|^2 - |L_1(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2 - 1}})|^2 +$$

$$+ p|L_1(\mathbb{Z}_{p^{\alpha_1 - 1}} \times \mathbb{Z}_{p^{\alpha_2 - 1}})|^2 = p^{2\alpha_1 - 1}\left[(2\alpha_2{-}2\alpha_1{+}1)p{-}2\alpha_2{+}2\alpha_1{+}1\right]$$

by Theorem 4.2 of [10].

In the case $k \geq 3$ the desired equality $CF_2(G) = 0$ follows by a direct calculation from (5) and **Theorem 4.3.** of [10], or by observing that G cannot be generated by two elements (and consequently cannot be written as a product of two cyclic subgroups). This completes the proof. ∎

In the second part of this section we will focus on computing the cyclic factorization number of other classes of finite groups starting with the dihedral groups

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1,\ yxy = x^{-1}\rangle,\ n \geq 3\,.$$

The subgroup structure of D_{2n} is the following: for every divisor d or n, D_{2n} possesses a subgroup isomorphic to $\mathbb{Z}_d$, namely $H_d = \langle x^{\frac{n}{d}}\rangle$, and $\frac{n}{d}$ subgroups isomorphic to D_{2d}, namely $K_d^i = \langle x^{\frac{n}{d}}, x^{i-1}y\rangle$, $i = 1, 2, ..., \frac{n}{d}$. We easily infer that

$$|L_1(H_d)| = \tau(d) \text{ and } |L_1(K_d^i)| = \tau(d) + d\,, \tag{6}$$

where $\tau(d)$ denotes the number of divisors of d. On the other hand, we know that

$$csd(H_d)=1 \text{ and } csd(K_d^i)=\begin{cases} \dfrac{\tau(d)(\tau(d)+d)+d(\tau(d)+1)}{(\tau(d)+d)^2}, & d \equiv 1 \,(\mathrm{mod}\, 2) \\[2ex] \dfrac{\tau(d)(\tau(d)+d)+d(\tau(d)+2)}{(\tau(d)+d)^2}, & d \equiv 0 \,(\mathrm{mod}\, 2) \end{cases} \tag{7}$$

by **Theorem 3.2.1.** of [14]. Also, the values of the Möbius function associated to $L(D_{2n})$ have been determined in **Lemma 19** of [1]:

$$\mu(H_d, D_{2n}) = \mu(D_{2\frac{n}{d}}) = -\frac{n}{d}\mu(\frac{n}{d}) \text{ and } \mu(K_d^i, D_{2n}) = \mu(\mathbb{Z}_{\frac{n}{d}}) = \mu(\frac{n}{d}) \tag{8}$$

for all divisors d of n and all $i = 1, 2, ..., \frac{n}{d}$. We are now able to complete the computation of $CF_2(D_{2n})$.

Theorem 3.2. *The cyclic factorization number of the dihedral group D_{2n}, $n \geq 3$, is given by the following equality:*

$$CF_2(D_{2n}) = 2n\,.$$

Proof. By using (6)-(8) in (4), one obtains

$$CF_2(D_{2n}) = \sum_{H \leq D_{2n}} csd(H)|L_1(H)|^2\mu(H, D_{2n}) =$$

$$= \sum_{d|n} csd(H_d)|L_1(H_d)|^2\mu(H_d, D_{2n}) + \sum_{d|n} \frac{n}{d}\, csd(K_d^1)|L_1(K_d^1)|^2\mu(K_d^1, D_{2n}) =$$

$$= \sum_{d|n} \tau(d)^2 \left(-\frac{n}{d}\mu(\frac{n}{d})\right) + \sum_{d|n} \frac{n}{d}\, \left[\tau(d)(\tau(d)+d) + d(\tau(d)+i_d)\right] \mu(\frac{n}{d}) =$$

$$= 2n \sum_{d|n} \tau(d)\mu(\frac{n}{d}) + n \sum_{d|n} i_d\mu(\frac{n}{d}),$$

where

$$i_d = \begin{cases} 1, & d \equiv 1 \pmod 2 \\ \\ 2, & d \equiv 0 \pmod 2 \end{cases}$$

We observe that

$$\sum_{d|n} \tau(d)\mu(\frac{n}{d}) = 1 \text{ and } \sum_{d|n} i_d\mu(\frac{n}{d}) = 0$$

and consequently

$$CF_2(D_{2n}) = 2n\,,$$

as desired. ■

Remark 3.3. An alternative and more facile way of computing $CF_2(D_{2n})$ uses its definition. Clearly, we have

$$L_1(D_{2n}) = \{H_d \mid d|n\} \cup \{K_1^i \mid i = 1, 2, ..., n\}.$$

We easily infer that D_{2n} is the product of two cyclic subgroups if and only if one of them is H_n and the other is of type K_1^i with $i = 1, 2, ..., n$. Hence $CF_2(D_{2n}) = 2n$.

The computation of the factorization numbers of the dihedral group D_{2n} and of the abelian p-group $G \cong \bigtimes_{i=1}^{k} \mathbb{Z}_{p^{\alpha_i}}$, $1 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_k$, will be extremely helpful in our study. These results will be used to find the cyclic factorization numbers of the following 5 groups:

- the generalized quaternion group

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1, yxy^{-1} = x^{-1}\rangle, n \ge 3$$

- the quasi-dihedral group

$$S_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = x^{2^{n-2}-1} \rangle, n \geq 4$$

- the modular p-group

$$M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{p^{n-2}+1} \rangle, n \geq 3$$

- the dicyclic group

$$Dic_{4n} = \langle a, \gamma \mid a^{2n} = 1, \gamma^2 = a^n, a^\gamma = a^{-1} \rangle,$$

- the generalized dicyclic group

$$Dic_{4n}(A) = \langle A, \gamma \mid \gamma^4 = e, \gamma^2 \in A \setminus \{e\}, g^\gamma = g^{-1}, \forall g \in A \rangle,$$

where A is an abelian group of order $2n$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_n$.

As we saw in the alternative proof concerning the cyclic factorization number of the dihedral group D_{2n}, it is important to know the cyclic subgroup structure of the group we study. Hence, it is useful to recall that the poset of cyclic subgroups of Q_{2^n} is

$$L_1(Q_{2^n}) = L(\langle x \rangle) \cup \{\langle x^k y \rangle \mid k = \overline{0, 2^{n-2} - 1}\}.$$

We are ready to compute the factorization number of the generalized quaternion group.

Theorem 3.4. *The cyclic factorization number of the generalized quaternion group Q_{2^n}, $n \geq 3$, is given by*

$$CF_2(Q_{2^n}) = \begin{cases} 6, & \text{if } n = 3 \\ 2^{n-1}, & \text{if } n \geq 4 \end{cases}.$$

Proof. By inspecting the subgroup lattice of Q_8, we infer that $CF_2(Q_8) = 6$. Let $n \geq 4$ be a positive integer. The unique minimal subgroup of the generalized quaternion group Q_{2^n} is its center $Z(Q_{2^n}) = \langle x^{2^{n-2}} \rangle$. Also, it is known that

$$\frac{Q_{2^n}}{Z(Q_{2^n})} \cong D_{2^{n-1}}.$$

Let H and K be two nontrivial cyclic subgroups of Q_{2^n} such that $HK = Q_{2^n}$. Then the two subgroups contain $Z(Q_{2^n})$ and by the above isomorphism, the cyclic factorization (H, K) of Q_{2n} corresponds to the cyclic factorization $(\frac{H}{Z(Q_{2^n})}, \frac{K}{Z(Q_{2^n})})$ of $D_{2^{n-1}}$. Also, it is important to remark that by quotiening non-cyclic subgroups of Q_{2^n} by the center of the group, we do not obtain corresponding cyclic subgroups of $D_{2^{n-1}}$. In fact, by this procedure, we always get a dihedral group or a direct product of two abelian groups, namely $\mathbb{Z}_2 \times \mathbb{Z}_2$. With this being checked, we are sure that there are no cyclic factorizations of $D_{2^{n-1}}$ that could correspond to non-cyclic factorization of Q_{2^n}. We add that it is obvious that we can not build another cyclic factorization of the generalized quaternion group using its trivial subgroup as a factor. Hence $CF_2(Q_{2^n}) = CF_2(D_{2^{n-1}}) = 2^{n-1}$ and our proof is complete. ∎

We will see that it may happen that by quotiening we can find a non-cyclic factorization of a group which corresponds to a cyclic factorization of its factor. The class of quasi-dihedral groups provides such an example. First of all, we recall that the cyclic subgroups of S_{2^n} are illustrated by the poset

$$L_1(S_{2^n}) = L(\langle x \rangle) \cup \{\langle x^{2k}y \rangle \mid k = \overline{0, 2^{n-2}-1}\} \cup \{\langle x^{2k+1}y \rangle \mid k = \overline{0, 2^{n-3}-1}\}.$$

Using this information, we can find the factorization number of the quasi-dihedral group.

Theorem 3.5. *The cyclic factorization number of the quasi-dihedral group S_{2^n}, $n \geq 4$, is given by*

$$CF_2(S_{2^n}) = 3 \cdot 2^{n-2}.$$

Proof. Besides the center $Z(S_{2^n}) = \langle x^{2^{n-2}} \rangle$, the quasi-dihedral group has other minimal subgroups contained in the set $M = \{\langle x^{2k}y \rangle \mid k = \overline{0, 2^{n-2}-1}\}$. Moreover any nontrivial subgroup of S_{2n} contains the center or it is contained in the set M. Let H and K be two cyclic subgroups such that $Z(S_{2^n}) \subseteq H$, $Z(S_{2^n}) \subseteq K$ and $HK = S_{2^n}$. In this case, to count the cyclic factorizations of S_{2^n}, we will use the isomorphism

$$\frac{S_{2^n}}{Z(S_{2^n})} \cong D_{2^{n-1}}.$$

This result implies that a cyclic factorization (H, K) of S_{2^n} corresponds to a cyclic factorization of $D_{2^{n-1}}$. However, the quasi-dihedral group S_{2^n} contains

2^{n-3} subgroups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and their images through the above isomorphism are cyclic subgroups of $D_{2^{n-1}}$ isomorphic to $\mathbb{Z}_2$. Each such subgroup is not contained in the maximal cyclic subgroup of $D_{2^{n-1}}$ and this leads us to $2 \cdot 2^{n-3} = 2^{n-2}$ cyclic factorizations of $D_{2^{n-1}}$ which correspond to non-cyclic factorizations of S_{2^n}. We remark that for other non-cyclic subgroups of the quasi-dihedral group S_{2^n}, we obtain non-cyclic subgroups of the dihedral group $D_{2^{n-1}}$ after quotiening by $Z(S_{2^n})$. It is clear that we can not build a cyclic factorization of S_{2^n} using two subgroups contained in M. Hence, the remaining option is to find cyclic factorizations (H, K), where $H \in M$ and K contains $Z(S_{2^n})$. Since, in this case, $H \cap K$ is trivial, we have only one choice for K, this being $K \cong \mathbb{Z}_{2^{n-1}}$, which is the only maximal cyclic subgroup of S_{2n}. By a simple computation, we get

$$CF_2(S_{2^n}) = CF_2(D_{2^{n-1}}) - 2^{n-2} + 2^{n-1} = 3 \cdot 2^{n-2},$$

as desired. ∎

The class of modular p-groups $M(p^n)$ provided interesting probabilistic aspects such as $sd(M(p^n)) = csd(M(p^n)) = 1$. The next purpose of this paper is to find an explicit formula for the cyclic factorization numbers of these groups.

Theorem 3.6. *The cyclic factorization number of the modular p-group M_{p^n}, $n \geq 3$, is given by*

$$CF_2(M(p^n)) = \begin{cases} 8, & \text{if } p = 2, n = 3 \\ p[(2n-4)(p-1) - p + 3] + 2p(p-1), & \text{if } p \geq 2, n \geq 4 \end{cases}.$$

Proof. Using **Theorem 3.2.** and the isomorphism $M(8) \cong D_8$, we infer that $CF_2(M(8)) = 8$. Let $p \geq 2$ and $n \geq 4$ be a prime number and a positive integer, respectively. By inspecting the lattice of subgroups of $M(p^n)$, we remark that this group has $p+1$ minimal subgroups isomorphic to $\mathbb{Z}_p$, one of them being the commutator subgroup $D(M(p^n)) = \langle x^{p^{n-2}} \rangle$ which is contained in each subgroup of higher order. Again, we will use an isomorphism to count the cyclic factorization number of the modular p-group. Indeed, it is known that

$$\frac{M(p^n)}{D(M(p^n))} \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}.$$

Let (H, K) be a cyclic factorization of $M(p^n)$ such that $D(M(p^n)) \subseteq H$ and $D(M(p^n)) \subseteq K$. Through the above isomorphism, there is a corresponding cyclic factorization of $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$, namely $(\frac{H}{D(M(p^n))}, \frac{K}{D(M(p^n))})$. There are $n-2$ non-cyclic proper subgroups of $M(p^n)$, these being isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^i}$, where $i = \overline{1, n-2}$. Quotiening these subgroups by $D(M(p^n))$, we obtain only one cyclic subgroup isomorphic to a minimal subgroup of $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$. This subgroup can be used to obtain $2p$ cyclic factorizations of $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$ since it is not contained in any maximal cyclic subgroup of this group. Hence, there are $2p$ cyclic factorizations of $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$ corresponding to non-cyclic factorizations of $M(p^n)$. The only method left to find other cyclic factorizations (H, K) of $M(p^n)$ is by using the p minimal subgroups different from $D(M(p^n))$ as a factor and the subgroups that contain $D(M(p^n))$ as the other factor. In this case $H \cap K$ is trivial, so the second factor must be one of the p maximal cyclic subgroups of $M(p^n)$ which are isomorphic to $\mathbb{Z}_{p^{n-1}}$, namely $\langle x \rangle$ and $\langle x^i y \rangle$, where $i = \overline{1, p-1}$. Adding up the numbers and using **Theorem 3.1.**, we have

$$CF_2(M(p^n)) = CF_2(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) - 2p + 2p^2 = p[(2n-4)(p-1) - p + 3] + 2p(p-1),$$

as stated above. ∎

A simplified formula is obtained for the modular 2-groups.

Corollary 3.7. *The cyclic factorization number of the modular 2-group $M(2^n)$, $n \geq 4$, is given by*

$$CF_2(M(2^n)) = 4n - 2.$$

Some probabilistic aspects of the dicyclic groups were studied in [15]. We know that for each divisor r of $2n$, the dicyclic group Dic_{4n} has only one subgroup isomorphic to $\mathbb{Z}_r$, namely $\langle a^{\frac{2n}{r}} \rangle$. Also, for each divisor s of n, Dic_{4n} possesses $\frac{n}{s}$ subgroups isomorphic to Dic_{4s}, namely $\langle a^{\frac{n}{s}}, a^{i-1}\gamma \rangle$, where $i = \overline{1, \frac{n}{s}}$. The poset of cyclic subgroups is

$$L_1(Dic_{4n}) = \{\langle a^{\frac{2n}{r}} \rangle \mid r | 2n\} \cup \{\langle a^{i-1}\gamma \rangle \mid i = \overline{1, n}\}.$$

We mention that, by definition, $Dic_4 \cong \mathbb{Z}_4$. Also, it is easy to see that $Dic_8 \cong Q_8$. Then $CF_2(Dic_4) = 5$ and $CF_2(Dic_8) = 6$. Finding the factorization numbers for all other dicyclic groups is our next purpose.

Theorem 3.8. *The cyclic factorization number of the dicyclic group Dic_{4n}, $n \geq 3$, is given by*

$$CF_2(Dic_{4n}) = \begin{cases} 4n, & \text{if } n \equiv 1(mod\ 2) \\ 2n, & \text{if } n \equiv 0(mod\ 2) \end{cases}.$$

Proof. It is known that $Z(Dic_{4n}) = \langle a^n \rangle$ which is isomorphic to $\mathbb{Z}_2$ and the following isomorphism holds

$$\frac{Dic_{4n}}{Z(Dic_{4n})} \cong D_{2n}.$$

We remark that by quotiening non-cyclic subgroups of Dic_{4n} we do not obtain cyclic subgroups of D_{2n}, so there will be no need to substract a number of factorizations from $CF_2(D_{2n})$ to find $CF_2(Dic_{4n})$, like we did in the case of other classes of groups. If $n = 2^m$, where $m \geq 2$ is a positive integer, since $Dic_{4n} \cong Q_{4n}$, we have $CF_2(Dic_{4n}) = CF_2(Q_{4n}) = 2n$ according to **Theorem 3.3.**. If n is even and is not a power of 2, a nontrivial subgroup H of Dic_{4n} contains the center of the group or it is a subgroup of $\mathbb{Z}_{2n}$ of odd order. Let (H, K) be a cyclic factorization of Dic_{4n} such that $Z(Dic_{4n}) \subseteq H$ and $Z(Dic_{4n}) \subseteq K$. Then there is a corresponding cyclic factorization of the dihedral group D_{2n}. If H is a nontrivial cyclic subgroup of odd order contained in the maximal subgroup $\mathbb{Z}_{2n}$ of Dic_{4n}, then we can use it to form a cyclic factorization if the other factor, K, is a cyclic subgroup which is not contained in $\mathbb{Z}_{2n}$. Looking at the poset $L_1(Dic_{4n})$, the only possible choices for K are the cyclic subgroups of order 4 from the set $\{\langle a^{i-1}\gamma \rangle \mid i = \overline{1, n}\}$. This leads us to $|H| = n$, which is false since n is even and $|H|$ is odd. Hence, if $n \equiv 1(mod\ 2)$, we obtain that

$$CF_2(Dic_{4n}) = CF_2(D_{2n}) = 2n.$$

The same reasoning is used if n is an odd number. The only change is that, in this case, we can find other cyclic factorizations of Dic_{4n} besides those corresponding to the ones of D_{2n}. Let K be one of the subgroups contained in the set $\{\langle a^{i-1}\gamma \rangle \mid i = \overline{1, n}\}$. Again, if H is a nontrivial cyclic subgroup of odd order contained in $\mathbb{Z}_{2n}$, then (H, K) is a cyclic factorization of Dic_{4n} if

$|H| = n$. According to the subgroup structure of the dicyclic group, there is only one such subgroup isomorphic to $\mathbb{Z}_n$. It follows that

$$CF_2(Dic_{4n}) = CF_2(D_{2n}) + 2n = 4n,$$

and our proof is complete. ∎

The last class of groups we study is formed by the generalized dicyclic groups $Dic_{4n}(A) = \langle A, \gamma \mid \gamma^4 = e, \gamma^2 \in A \setminus \{e\}, g^\gamma = g^{-1}, \forall g \in A \rangle$, where $A \cong \mathbb{Z}_2 \times \mathbb{Z}_n$. Let a and b be the generators of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_2$, respectively. Since $\gamma^2 \in A \setminus \{e\}$ and $\gamma^4 = e$, we infer that $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$. The subgroup strucute and explicit formulas for the (cyclic) subgroup commutativity degree of $Dic_{4n}(A)$ are provided by [15] and our final purpose in this paper is to compute its cyclic factorization number.

Theorem 3.9. *Let $A = \mathbb{Z}_2 \times \mathbb{Z}_n$ be an abelian group, where $n = 2^m m'$, $m \in \mathbb{N}^*$ and m' is a positive odd integer. Then,*

(i) if $m = 1, m' \neq 1$ and $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$ or if $m \geqslant 2, m'$ is any odd number and $\gamma^2 \in \{b, a^{\frac{n}{2}}\}$, the cyclic factorization number of the generalized dicyclic group $Dic_{4n}(A)$ is

$$CF_2(Dic_{4n}(A)) = 4n.$$

(ii) if $m \geqslant 2, m' \neq 1$ and $\gamma^2 = a^{\frac{n}{2}}$, the cyclic factorization number of the generalized dicylic group $Dic_{4n}(A)$ is

$$CF_2(Dic_{4n}(A)) = 0.$$

(iii) if $m \geqslant 2, m' = 1$ and $\gamma^2 = a^{\frac{n}{2}}$, the generalized dicyclic group $Dic_{4n}(A)$ is isomorphic to the direct product $\mathbb{Z}_2 \times Q_{2^{m+1}}$ and its cyclic factorization number is

$$CF_2(\mathbb{Z}_2 \times Q_{2^{m+1}}) = 0.$$

(iv) if $m = m' = 1$ and $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$, the generalized dicyclic group $Dic_{4n}(A)$ is isomorphic to the abelian group $\mathbb{Z}_2^3$ and its cyclic factorization number is

$$CF_2(\mathbb{Z}_2^3) = 0.$$

Proof. The cyclic subgroups of $Dic_{4n}(A)$ are those contained in the abelian group A and the subgroups contained in the set $C_1 = \{\langle a^{i-1}\gamma\rangle \mid i = \overline{1,n}\}$ if $\gamma^2 \in \{b, a^{\frac{n}{2}}b\}$, respectively in the set $C_2 = \{\langle (ab)^{i-1}\gamma\rangle \mid i = \overline{1,n}\}$ if $\gamma^2 = a^{\frac{n}{2}}$.

(i) Let $m \geqslant 2, m'$ a positive odd number and $\gamma^2 \in \{b, a^{\frac{n}{2}}b\}$. Since $|Dic_{4n}(A)| = 4n$ and, in this case, $4n \geqslant 16$ we can not form a cyclic factorization using only cyclic subgroups contained in C_1 which are of order 4 and do not intersect trivially. Hence a cyclic factorization (H, K) of $Dic_{4n}(A)$ is formed by a cyclic subgroup H of A and a cyclic subgroup K belonging to C_1. This leads us to $|H| \geq n$ and since $A \cong \mathbb{Z}_2 \times \mathbb{Z}_n$, it is clear that $|H| = n$. In other words, H is a maximal cyclic subgroup of A. It is easy to see that A has 2 maximal cyclic subgroups, namely $\langle a\rangle$ and $\langle ab\rangle$. We remark that the intersection of each one of these subgroups with any subgroup contained in C_1 is trivial. Hence, $CF_2(Dic_{4n}(A)) = 4n$. If $m = 1, m' \neq 1$ and $\gamma^2 \in \{b, a^{\frac{n}{2}}b\}$, the reasoning is similar, the only change being that A contains 3 maximal cyclic subgroups, namely $\langle a\rangle, \langle ab\rangle$ and $\langle a^2b\rangle$. It is easy to see that only two of these subgroups trivially intersect the n subgroups belonging to C_1 as we choose γ^2 to be b or $a^{\frac{n}{2}}b$, so we obtain again that $CF_2(Dic_{4n}(A)) = 4n$. If $m = 1, m' \neq 1$ and $\gamma^2 = a^{\frac{n}{2}}$, the cyclic factorizations of $Dic_{4n}(A)$ are formed by the cyclic subgroups contained in C_2 and two of the three maximal cyclic subgroups contained in A, namely $\langle ab\rangle$ and $\langle a^2b\rangle$. Therefore, in all the mentioned cases, the cyclic factorization number of the generalized dicyclic group $Dic_{4n}(A)$ is

$$CF_2(Dic_{4n}(A)) = 4n.$$

(ii) Let $m \geqslant 2, m' \neq 1$ and $\gamma^2 = a^{\frac{n}{2}}$. The only possible cyclic factorizations (H, K) of $Dic_{4n}(A)$ are formed by the factors $H \in L_1(A)$ such that $|H| = n$ and $K \in C_2$. We saw that A has two maximal cyclic subgroups, but since their intersection with the n cyclic subgroups contained in C_2 is $\{e, \gamma^2\}$, it is clear that

$$CF_2(Dic_{4n}(A)) = 0.$$

(iii) Let $m \geqslant 2, m' = 1$ and $\gamma^2 = a^{\frac{n}{2}}$. For more details about the mentioned isomorphism, we refer the reader to [15]. Denote by x, y, b_1 and e_1 the elements $a\langle\gamma^2\rangle, \gamma\langle\gamma^2\rangle, b\langle\gamma^2\rangle$ and $e\langle\gamma^2\rangle$, respectively. We get the following isomorphism

$$\frac{\mathbb{Z}_2 \times Q_{2^{m+1}}}{\langle\gamma^2\rangle} = \langle b_1\rangle \times \langle x, y \mid x^{2^m} = y^2 = e_1, x^y = x^{-1}\rangle \cong \mathbb{Z}_2 \times D_{2n'},$$

where $n' = \frac{n}{2} = 2^{m-1}$. This isomorphism implies that the group $\mathbb{Z}_2 \times D_{2n'}$ contains the subgroup lattice of $A' = \langle b_1, x \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{n'}$ obtained by quotiening A by $\langle \gamma^2 \rangle$. In the same paper, we saw that besides the cyclic subgroups contained in A', the group $\mathbb{Z}_2 \times D_{2n'}$ has other $2n'$ cyclic subgroups of order 2, namely $\langle x^{i-1} y \rangle$ and $\langle x^{i-1} y b_1 \rangle$, where $i = \overline{1, n'}$. Since a factor H of a possible cyclic factorization (H, K) of $\mathbb{Z}_2 \times D_{2n'}$ is one of the $2n'$ above mentioned cyclic subgroups of order 2, the other factor K must be a cyclic subgroup contained in A' such that $|K| \geq 2n'$. However, $A' \cong \mathbb{Z}_2 \times \mathbb{Z}_{n'}$, so there is no such K. It follows that

$$CF_2(\mathbb{Z}_2 \times D_{2n'}) = CF_2(\mathbb{Z}_2 \times D_{2^m}) = 0.$$

A nontrivial subgroup of $\mathbb{Z}_2 \times Q_{2^{m+1}}$ contains $\langle \gamma^2 \rangle = \langle a^{n'} \rangle$ or it is one of the minimal subgroups $\langle b \rangle$ or $\langle a^{n'} b \rangle$. Assume that (H, K) is a cyclic factorization of $\mathbb{Z}_2 \times Q_{2^{m+1}}$ such that $\langle \gamma^2 \rangle \subseteq H$ and $\langle \gamma^2 \rangle \subseteq K$. Then there exists a corresponding cyclic factorization of $\mathbb{Z}_2 \times D_{2n'}$, which is false since we proved that $CF_2(\mathbb{Z}_2 \times D_{2n'}) = 0$. It is clear that other possible cyclic factorization (H, K) of $\mathbb{Z}_2 \times Q_{2^{m+1}}$ is formed by a factor H being $\langle b \rangle$ or $\langle a^{n'} b \rangle$ and a factor K that contains $\langle \gamma^2 \rangle$. Since $|H| = 2$, it follows that the order of the cyclic subgroup K is 2^{m+1} or 2^{m+2}. However, the group $\mathbb{Z}_2 \times Q_{2^{m+1}}$ do not have any element that could generate such a cyclic subgroup. Therefore, we proved that

$$CF_2(\mathbb{Z}_2 \times Q_{2^{m+1}}) = 0.$$

(iv) Let $m = m' = 1$ and $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}} b\}$. It is straightforward to check the isomorphism $Dic_{4n}(A) \cong \mathbb{Z}_2^3$. Also, by **Theorem 3.1.**, we have

$$CF_2(\mathbb{Z}_2^3) = 0,$$

as desired. ∎

4 Conclusions and further research

All our previous results show that the cyclic factorization number constitutes an interesting computational aspect of finite groups. It is clear that the study started here can successfully be extended to other classes of finite groups. This will surely be the subject of some further research.

We end our paper by formulating several open problems concerning this topic.

Problem 4.1. Compute explicitly the cyclic factorization number of a finite Zassenhaus metacyclic group (see e.g. [4], I).

Problem 4.2. Determine the class of finite groups/p-groups G for which $CF_2(G) \neq 0$.

Problem 4.3. Study other connections between the cyclic factorization number and the factorization number of a finite group.

Problem 4.4. Given a finite non-nilpotent group G, describe the manner in which $CF_2(G)$ depends on the cyclic factorization numbers of the Sylow subgroups of G.

Problem 4.5. What can be said about two finite groups of order n having the same cyclic factorization number?

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